

Region unknotting number of 2-bridge knots

Vikash. S, Madeti. P

Department of Mathematics, IIT Ropar, Rupnagar - 140001, India.

Abstract

In this paper, we discuss the region unknotting number of different classes of 2-bridge knots. In particular, we provide region unknotting number for the classes of 2-bridge knots whose Conway notation is $C(m, n)$, $C(m, 2, m)$, $C(m, 2, m \pm 1)$ and $C(2, m, 2, n)$. By generalizing, we also provide a sharp upper bound for all the remaining classes of 2-bridge knots.

1. Introduction

In [1], A. Shimizu introduced a new local transformation on link diagrams and named it as *region crossing change*. In [1, 5], it was proved that this new local transformation is an unknotting operation for a knot or a proper link. Here a region crossing change at a region R of a knot diagram D is defined to be the crossing changes at all the crossing points on ∂R . The region unknotting number $u_R(D)$ of a knot diagram D is the minimum number of region crossing changes required to transform D into a diagram of the trivial knot without Reidemeister moves. The region unknotting number $u_R(K)$ of K is defined to be the minimal $u_R(D)$ taken over all minimal crossing diagrams D of K . In [5], Z. Cheng proved that region crossing change for a link is an unknotting operation if and only if the link is proper.

Many knot theorists studied different unknotting operations like \sharp -operation [11], δ -operation [10], 3-gon operation [7], $H(n)$ -operation [9] and n -gon [8] operations. In [7] Y. Nakanishi proved that a δ -unknotting operation can be obtained from a finite sequence of 3-gon moves. In [8], H. Aida generalized 3-gon moves to n -gon moves and proved that an n -gon move is an unknotting operation.

It is interesting to observe that both \sharp -operation and n -gon moves are special cases of region crossing change. Finding region unknotting number for different knots is a challenging problem. In [1], A. Shimizu showed that for a twist knot K , $u_R(K) = 1$ and for torus knots of type $K(2, 4m \pm 1)$, $u_R(K(2, 4m \pm 1)) = m$, where $m \in \mathbb{Z}^+$. In [6], we provided a sharp upper bound for region unknotting number of torus knots.

In this paper, we provide region unknotting number for all those 2-bridge knots whose Conway's notation is $C(m, n)$, $C(m, 2, n)$, $C(m, 2, m \pm 1)$ and $C(2, m, 2, n)$. We also discuss some bounds on region unknotting number for other 2-bridge knot classes. Since minimal crossing diagrams are required to find region unknotting number of knots, we mainly look for all the 2-bridge knot diagrams with minimum crossings. In this context, it is required to observe that all 2-bridge knots are prime [14] and alternating [12]. Specifically, using Tait's third conjecture, which is true

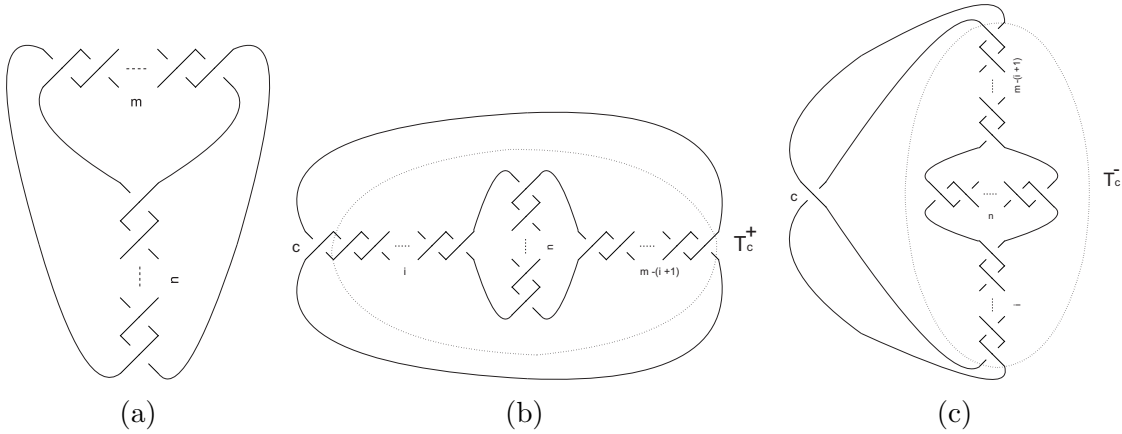


Figure 1: Minimal diagram for 2-bridge knot $C(m, n)$

[13], one can obtain all the minimal diagrams of a prime reduced alternating knot K from a minimal crossing diagram of K by performing finite number of flypings. In [1], A. Shimizu provided a method to find all possible minimal crossing diagrams of a prime alternating knot. Here our concentration is only on the 2-bridge knots.

Based on the method provided in [1], we can show a 2-bridge knot $C(m, n)$, where $(m, n \neq 0 \in \mathbb{Z}, mn > 0)$, has only one minimal crossing diagram on S^2 . Let D , as in Figure 1(a), be a minimal crossing diagram of a 2-bridge knot $C(m, n)$. For every crossing point c in integer tangle t_m (horizontal tangle having m half twists), T_c^+ and T_c^- are shown in Figure 1(b) and 1(c), respectively. To get non-trivial flyping and hence non equivalent minimal diagrams of $C(m, n)$, T_c^+ and T_c^- should not satisfy any of the following three conditions:

1. the tangle T_c^ϵ is not a tangle sum ($\epsilon = +, -$)
2. the tangle T_1 or T_2 is an integer 2-tangle
3. the tangles T_1 and T_2 satisfy $T_{1hv} = T_1$ and $T_{2v} = T_2$, or $T_{1v} = T_1$ and $T_{2hv} = T_2$.

Observe that the tangle T_c^+ is sum of two tangles T_1 and T_2 , where either T_1 is t_i and T_2 is tangle sum of t'_n (vertical tangle having n half twists) and $t_{m-(i+1)}$ or T_1 is tangle sum of t_i and t'_n and T_2 is $t_{m-(i+1)}$ (where $0 \leq i \leq m-1$). The tangle T_c^- is not a tangle sum. Then T_c^+ and T_c^- satisfy the cases (2) and (1) respectively. Therefore we can not perform non-trivial flyping on any c and T_c^ϵ , where c is a crossing in m -tangle. Since $C(m, n) \sim C(n, m)$, same happens for any crossing c from vertical tangle t'_n . Hence, 2-bridge knot $C(m, n)$ has only one minimal diagram.

In Section 2, we provide region unknotting number for 2-bridge knot classes whose Conway's notation is $C(m, n)$, $C(m, 2, m)$ and $C(m, 2, m \pm 1)$. Also we provide an upper bound for region unknotting number for all 2-bridge knots. In Section 3, we provide Arf invariant for 2-bridge knots (not links) whose Conway's notation is $C(m, n)$ and $C(m, p, n)$.

2. Region unknotting number for 2-bridge knots

In this section we provide region unknotting number of 2-bridge knot K whose Conway notation is $C(m, n)$ by showing $u_R(K) = u_R(C(m, n))$. Region unknotting number of 2-bridge knots whose Conway notation is $C(m, 2, m)$ and $C(m, 2, m \pm 1)$ is shown to be one. Also we give upper bound for 2-bridge knot classes whose Conway notation is $C(m, 2, n)$, $C(m, p, n)$, $C(c_1, c_2, \dots, c_n)$ where c_{2k+1} is even and $C(c_1, c_2, \dots, c_n)$ where c_{2k} is even and n is even. At last, a general upper bound for region unknotting number for all 2-bridge knots is also provided.

The key idea to ensure the region unknotting number is that in a 2-bridge knot $C(c_1 c_2 \dots c_n)$ each tangle c_i is a $(2, q)$ type toric braid and by [6], region unknotting number of $(2, q)$ type torus knot or proper link is $\lfloor \frac{q+2}{4} \rfloor$. Hence, to convert an integer 2-tangle t_n or t'_n into 0 or ∞ tangle, respectively, we need to make atleast $\lfloor \frac{n+2}{4} \rfloor$ region crossing changes. Throughout this paper, we consider only those 2-bridge knots which are either knots or proper links. Observe that in $C(m, n)$, if both m and n are odd and $m + n \not\equiv 0 \pmod{4}$, then 2-bridge knot $C(m, n)$ is not proper.

Theorem 2.1. *Let K be a 2-bridge knot/proper link whose Conway's notation is $C(m, n)$. Then we have the following:*

1. if m, n are even, then $u_R(K) = \lfloor \frac{\min\{m, n\} + 2}{4} \rfloor$,
2. if m even, n odd, then $u_R(K) = \lfloor \frac{m+2}{4} \rfloor$,
3. if m odd, n even, then $u_R(K) = \lfloor \frac{n+2}{4} \rfloor$,
4. if m, n are odd, then $u_R(K) = \frac{m+n}{4}$.

PROOF. Since K is a 2-bridge knot with Conway notation $C(m, n)$, the only minimal diagram for K is as shown in Figure 2(a). From Figure 2(a), it is clear that this minimal diagram of K has total $m + n + 2$ regions, out of which the regions $R_1, R_{m+1}; R'_1, R'_{n+1}$; and the remaining $m + n - 2$ regions have $n + 1; m + 1$; and 2 crossings respectively on their boundaries.

To get trivial knot diagram from $C(m, n)$, we need to make region crossing changes such that sum of signs of crossings in either horizontal or vertical tangle become 0. In other words, to transform $C(m, n)$ to unknot by region crossing changes, we need to reduce either m or n to 0. In this process of selection of regions, observe that a region crossing change at any one of R_1, R_{m+1} or R'_1, R'_{n+1} in $C(m, n)$ will reduce m to $m - 2$ or n to $n - 2$ respectively and hence at each step, the absolute value of the sum of signs of crossings of either horizontal or vertical tangle reduce by 2. But the region crossing change at any other region will reduce sum of signs of crossings of either horizontal or vertical tangle by 4.

Since the choice of regions is based on the values of m and n , here we provide region unknotting number of $C(m, n)$ for all possible cases of m and n .

Case (i) If both m and n are even:

Without loss of generality assume $n \leq m$. If $n \equiv 0 \pmod{4}$, then make region crossing changes at any non-consecutive $\frac{n}{4}$ regions among $R'_j (2 \leq j \leq n)$. These region crossing changes reduce the absolute value of sum of signs of crossings of vertical tangle to zero i.e., the diagram $C(m, n)$

transforms to a diagram of $C(m, 0)$, which is m times twisted unknot. Hence $u_R(K) \leq \frac{n}{4}$. Since it is not possible to reduce a t'_n tangle to ∞ tangle with less than $\frac{n}{4}$ region crossing changes, $u_R(K) = \frac{n}{4} = \lfloor \frac{n+2}{4} \rfloor$.

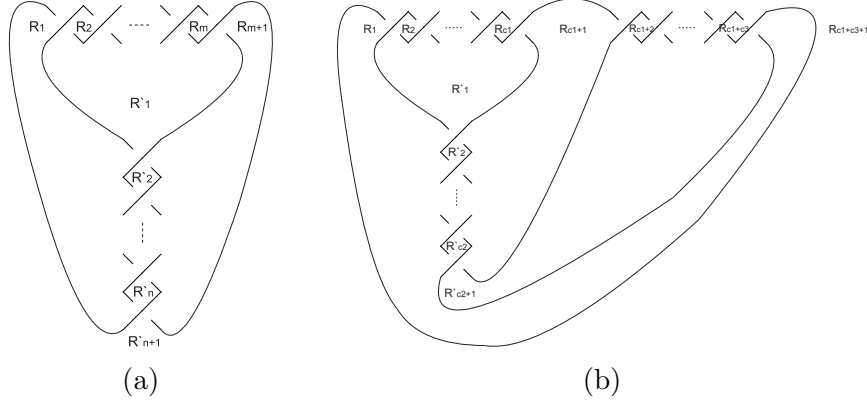


Figure 2: Region Data for 2-bridge knot

If $n \not\equiv 0 \pmod{4}$, then region crossing changes at any non-consecutive $\frac{n-2}{4}$ regions of $R'_j (3 \leq j \leq n)$, transforms the diagram $C(m, n)$ into $C(m, 2)$. Note that each of these region crossing change reduces the sum of signs of crossings by 4. Then region crossing change at any non-consecutive $\frac{n-2}{4}$ regions of $R'_j (3 \leq j \leq n)$ and R'_1 in $C(m, n)$ results in a trivial knot diagram. It is easy to observe that these are the minimum number of regions required to convert $C(m, n)$ to a trivial knot diagram. Hence, $u_R(K) = \frac{n-2}{4} + 1 = \lfloor \frac{n+2}{4} \rfloor$

Case (ii) If m is even, n is odd:

Observe that by making region crossing changes at any non-consecutive $\lfloor \frac{n+2}{4} \rfloor$ regions from $R'_j (2 \leq j \leq n)$ in t'_n tangle, the resultant diagram will be a diagram of either $(2, m+1)$ or $(2, m-1)$ torus knot. Since $u_R(2, q) = \lfloor \frac{q+2}{4} \rfloor$, $u_R(K) \leq \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{m+3}{4} \rfloor$ or $u_R(K) \leq \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{m+1}{4} \rfloor$. Note that, here we make region crossing changes in both t'_n and t_m tangles. But if we first make region crossing changes in t_m as in case (i), we get $u_R(K) \leq \lfloor \frac{m+2}{4} \rfloor$.

Since these are the only possibilities of choices of regions to convert K to an unknot and since $\lfloor \frac{m+2}{4} \rfloor \leq \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{m+3}{4} \rfloor$ and $\lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{m+1}{4} \rfloor$, we have $u_R(K) = \lfloor \frac{m+2}{4} \rfloor$.

Case (iii) If n is even and m is odd:

Proof is similar to case when m is even and n is odd as $C(m, n) = C(n, m)$. In this case $u_R(K) = \lfloor \frac{n+2}{4} \rfloor$.

Case (iv) If both m and n are odd:

It is easy to observe that neither m nor n separately can reduce to 0. Using the same procedure as in case 2, if $n \equiv 1 \pmod{4}$, we get $u_R(K) = \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{m+3}{4} \rfloor = \frac{m+n}{4}$.

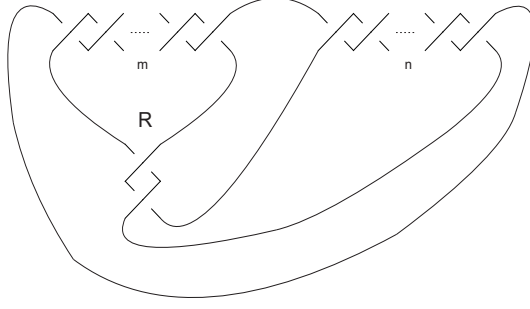


Figure 3: $C(m, 2, n)$

If $n \equiv -1 \pmod{4}$, then $u_R(K) = \lfloor \frac{n+2}{4} \rfloor + \lfloor \frac{m+1}{4} \rfloor = \frac{m+n}{4}$. Hence $u_R(K) = \frac{m+n}{4}$. \square

To provide region unknotting number for 2-bridge knots of type $C(m, 2, m)$, $C(m, 2, m \pm 1)$, first we provide an upper bound for region unknotting number for a general class of 2-bridge knot whose Conway's notation is $C(m, 2, n)$. Note that the 2-bridge knot $C(m, 2, n)$ is a 2-component link $L = K_1 \cup K_2$ iff $m \equiv n \pmod{2}$. Also $lk(K_1, K_2) = \begin{cases} \frac{m+n}{2} & \text{if both } m \text{ and } n \text{ are even;} \\ \frac{m+n+2}{2} & \text{if both } m \text{ and } n \text{ are odd.} \end{cases}$

It is easy to calculate that, in both the cases, link L will be proper iff $m \equiv n \pmod{4}$. For 2-bridge knots and proper links $C(m, 2, n)$, we have the following upper bound.

Theorem 2.2. *For 2-bridge knot K with Conway's notation $C(m, 2, n)$,*

$$u_R(K) \leq \left\lfloor \frac{|m-n|+2}{4} \right\rfloor + 1.$$

PROOF. Observe that after a region crossing change at region R as in Figure 3, the resultant diagram will be a diagram of $(2, n-m)$ -type torus knot/link. Since region unknotting number for $(2, n-m)$ torus knot/link is $\left\lfloor \frac{|m-n|+2}{4} \right\rfloor$, it is easy to observe that region crossing changes at any non-consecutive $\left\lfloor \frac{|m-n|+2}{4} \right\rfloor$ regions in t_m (if $m > n$) or in t_n (if $n > m$) together with region crossing change at R in $C(m, 2, n)$ provide trivial knot diagram. Thus

$$u_R(K) \leq \left\lfloor \frac{|m-n|+2}{4} \right\rfloor + 1.$$

\square

Corollary 2.1. *Region unknotting number for 2-bridge knot/link $C(m, 2, m)$ and $C(m, 2, m \pm 1)$ is one.*

PROOF. It is clear from Theorem 2.2 that if $n = m$ or $m \pm 1$ then $\left\lfloor \frac{|m-n|+2}{4} \right\rfloor = 0$. Hence $u_R(C(m, 2, n)) = 1$.

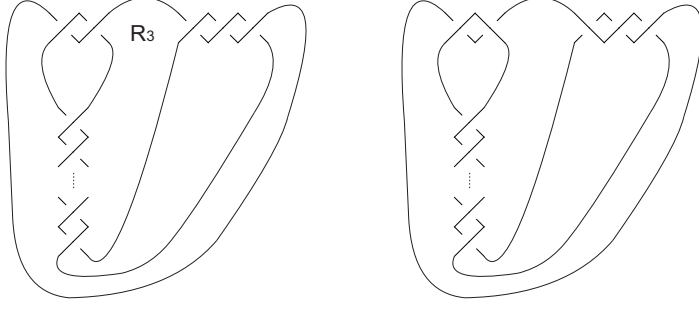


Figure 4: $C(2, p, 3)$

In case of 2-bridge link $C(m, p, n)$, we consider different cases depending on the values of m, n and p . Note that $C(m, p, n)$ is not proper in the following cases:

1. when both m and n are even and $m \not\equiv n \pmod{4}$
2. when both m and n are odd and p is even and $m + n + p \equiv 2 \pmod{4}$.

Theorem 2.3. *For 2-bridge knot K with Conway's notation $C(m, p, n)$, where either m or n is even,*

$$u_R(K) \leq \left\lfloor \frac{m + n + 2}{4} \right\rfloor.$$

PROOF. Without loss of generality, assume that m is even. After region crossing changing at the regions $\langle R_3, R_7, \dots, R_{3+4 \cdot \lfloor \frac{m-2}{4} \rfloor} \rangle$, the resultant diagram is a diagram of either $(2, n)$ or $(2, n - 2)$ type torus knot based on whether $m \equiv 0 \pmod{4}$ or $m \equiv 2 \pmod{4}$. Since region unknotting number for $(2, q)$ type torus knot is $\lfloor \frac{q+2}{4} \rfloor$, the region crossing changes at $\langle R_3, R_7, \dots, R_{3+4 \cdot \lfloor \frac{m-2}{4} \rfloor}, \dots, R_{3+4 \cdot \lfloor \frac{m+n-2}{4} \rfloor} \rangle$ regions transform $C(m, p, n)$ to a diagram of trivial knot. Hence the number of region crossing changes to unknot $C(m, p, n)$ is either $\lfloor \frac{m+2}{4} \rfloor + \lfloor \frac{n+2}{4} \rfloor$ or $\lfloor \frac{m+2}{4} \rfloor + \lfloor \frac{n}{4} \rfloor$ based on whether $m \equiv 0 \pmod{4}$ or $m \equiv 2 \pmod{4}$. Hence

$$u_R(K) \leq \left\lfloor \frac{m + n + 2}{4} \right\rfloor.$$

□

Remark 1. From Theorem 2.3, it is easy to observe that $u_R(C(2, p, 3)) = 1$ for any p . As shown in Figure 4, region crossing change at R_3 , results in a trivial knot diagram.

Theorem 2.4. *For 2-bridge knot/proper link K with Conway's notation $C(m, p, n)$, where p is even, we have*

1. when $p \equiv 2 \pmod{4}$,

$$u_R(K) \leq \left\lfloor \frac{|m - n| + 2}{4} \right\rfloor + \left\lfloor \frac{p + 2}{4} \right\rfloor$$

2. when $p \equiv 0 \pmod{4}$,

$$u_R(K) \leq \begin{cases} \left\lfloor \frac{m+n+2}{4} \right\rfloor & \text{if either } m \text{ or } n \text{ is even} \\ \frac{m+n+p}{4} & \text{if both } m \text{ and } n \text{ are odd} \end{cases}.$$

PROOF. Case (i) Observe that after region crossing changes at $\langle R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-2}{4} \rfloor} \rangle$ regions in $C(m, p, n)$, the resultant diagram is a diagram of $(2, m-n)$ -type torus knot/link. Since region unknotting number for $(2, n-m)$ torus knot/link is $\left\lfloor \frac{|m-n|+2}{4} \right\rfloor$, by selecting any non-consecutive $\left\lfloor \frac{|m-n|+2}{4} \right\rfloor$ regions in t_m (if $m > n$) or in t_n (if $n > m$) results in a trivial knot diagram. Thus

$$u_R(K) \leq \left\lfloor \frac{|m-n|+2}{4} \right\rfloor + \left\lfloor \frac{p+2}{4} \right\rfloor.$$

Case (ii) If either m or n is even, then proof directly follows from Theorem 2.3. When both m and n are odd, it is easy to observe that after region crossing changes at $\langle R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-2}{4} \rfloor} \rangle$ regions, the resultant diagram is a diagram of $(2, m+n)$ -type torus knot/link. Since region unknotting number for $(2, m+n)$ torus knot/link is $\left\lfloor \frac{m+n+2}{4} \right\rfloor$, region crossing changes at $\langle R_3, R_7, \dots, R_{3+4 \cdot \lfloor \frac{m+n-2}{4} \rfloor}, R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-2}{4} \rfloor} \rangle$ in $C(m, p, n)$ transform $C(m, p, n)$ to a diagram of trivial knot. Observe that the number of region crossing changes is equal to $\left\lfloor \frac{m+n+2}{4} \right\rfloor + \left\lfloor \frac{p+2}{4} \right\rfloor$. Since, in this case, 2-bridge link $C(m, p, n)$ is proper iff $m+n \equiv 0 \pmod{4}$, we have $\left\lfloor \frac{m+n+2}{4} \right\rfloor = \frac{m+n}{4}$. Thus

$$u_R(K) \leq \frac{m+n+p}{4}.$$

□

In $C(m, p, n)$, if p is odd and either m or n is even, by Theorem 2.3, we have $u_R(K) \leq \left\lfloor \frac{m+n+2}{4} \right\rfloor$. In case when both m and n are odd, take $k = \min\{m, n\}$. If either one of m or n is odd then consider k to be that integer which is odd. In the following theorem, consider k as defined above.

Theorem 2.5. For 2-bridge knot K with Conway's notation $C(m, p, n)$, where p is odd and

1. if $p+k \equiv 0 \pmod{4}$, then

$$u_R(K) \leq \frac{p+k}{4}$$

2. if $p+k \equiv 2 \pmod{4}$, then

$$u_R(K) \leq \frac{p+k+2}{4}.$$

PROOF. Case (i) If $p+k \equiv 0 \pmod{4}$, then based on $k = m$ or n , we make region crossing changes at either $\langle R_3, R_7, \dots, R_{3+4 \cdot \lfloor \frac{n-2}{4} \rfloor}, R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-2}{4} \rfloor} \rangle$ or $\langle R_{m+3}, R_{m+7}, \dots, R_{m+3+4 \cdot \lfloor \frac{n-2}{4} \rfloor}, R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-2}{4} \rfloor} \rangle$ regions. After these region crossing changes we get a diagram of trivial knot. In any case, the resultant diagram is equivalent to diagram of $C(\pm 1, \mp 1, n) = C(0, n)$ or $C(m, \pm 1, \mp 1) = C(m, 0)$, which is a trivial knot. Observe that the number of region crossing

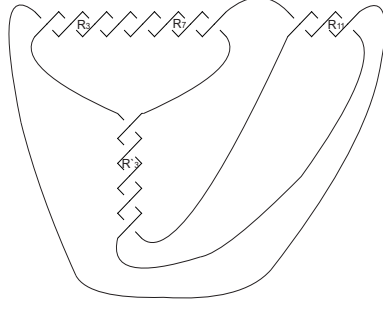


Figure 5: Region unknotting number for $C(8, 5, 3)$

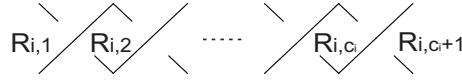


Figure 6: Region Data for integer tangle t_{c_i}

changes is equal to $\lfloor \frac{k+2}{4} \rfloor + \lfloor \frac{p+2}{4} \rfloor = \frac{p+k}{4}$ and hence, $u_R(K) \leq \frac{p+k}{4}$.

Case (ii) If $p+k \equiv 2 \pmod{4}$, then we have either both p and k are $\equiv 1 \pmod{4}$ or both p and k are $\equiv 3 \pmod{4}$. If both p and k are $\equiv 1 \pmod{4}$, then based on $k = m$ or n , we make region crossing changes at $\langle R_3, R_7, \dots, R_{3+4 \cdot \lfloor \frac{n-2}{4} \rfloor}, R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-2}{4} \rfloor} \rangle$ or $\langle R_{m+3}, R_{m+7}, \dots, R_{m+3+4 \cdot \lfloor \frac{n-2}{4} \rfloor}, R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-2}{4} \rfloor} \rangle$ regions. In case if both p and k are $\equiv 3 \pmod{4}$, then based on $k = m$ or n , we make region crossing changes at $\langle R_3, R_7, \dots, R_{3+4 \cdot \lfloor \frac{n-2}{4} \rfloor}, R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-4}{4} \rfloor} \rangle$ or $\langle R_{m+3}, R_{m+7}, \dots, R_{m+3+4 \cdot \lfloor \frac{n-2}{4} \rfloor}, R'_3, R'_7, \dots, R'_{3+4 \cdot \lfloor \frac{p-4}{4} \rfloor} \rangle$ regions. In case if $p = 3$, then we make no region region crossing changes in t'_p . After these region crossing changes, we get a diagram of a 2-bridge knot $C(m, 2)$. Then one more region crossing change at R'_1 transforms $C(m, p, n)$ to an unknot. Observe that the number of region crossing changes is equal to $\lfloor \frac{k+2}{4} \rfloor + \lfloor \frac{p+2}{4} \rfloor + 1$ or $\lfloor \frac{k+2}{4} \rfloor + \lfloor \frac{p}{4} \rfloor + 1$ respectively. Thus $u_R(K) \leq \frac{p+k+2}{4}$. \square

Remark 2. Note that in case of those 2-bridge knots which occurs in more than one category, we consider upper bound for region unknotting number to be minimum of the upper bounds from all undertaken categories. For example 2-bridge knot $C(8, 5, 3)$ satisfies the hypothesis of Theorem 2.3 and Theorem 2.5. By Theorem 2.3, $u_R(K) \leq 3$, as region crossing changes at R_3, R_7 and R_{11} regions in Figure 5, transforms $C(8, 5, 3)$ into trivial knot and by Theorem 2.5, $u_R(K) \leq 2$, as region crossing changes at R_{11} and R'_3 makes it unknot. So $u_R(C(8, 5, 3)) \leq 2$.

A general upper bound for region unknotting number is given for all 2-bridge knot/proper links. Due to generality, here we consider regions of each integer tangle (t_{c_i} or t'_{c_i}) as $R_{i,1}, R_{i,2}, \dots, R_{i,c_i+1}$ as in Figure 6. Note that region R_{i,c_i+1} of tangle t_{c_i} is same as the region $R_{i+2,1}$ of $t_{c_{i+2}}$. Similarly the region R_{i,c_i+1} of tangle t'_{c_i} is same as the region $R_{i+2,1}$ of $t'_{c_{i+2}}$ respectively.

To provide an upper bound for $C(c_1, c_2, \dots, c_n)$, first we construct a subset L of $2\mathbb{N}$ as follows:

- for $j = 2$, if $c_1 \equiv 0 \pmod{2}$ then $2 \in L$.
- for next even integer $j = 4$, if $2 \notin L$ and $c_1 + c_2 + c_3 \equiv 0 \pmod{2}$ then $4 \in L$. If $2 \in L$ and $c_1 + c_3 \equiv 0 \pmod{2}$ then $4 \in L$.
- Continuing in the same way, any even integer $j(\leq n) \in L$ if

$$\sum_{\substack{i < j \\ i \notin L}} c_i \equiv 0 \pmod{2}.$$

In the following theorem, we will observe that by selectively choosing some region crossing changes, there is no need to make any region crossing change in t'_{c_j} , where $j \in L$, to transform $C(c_1, c_2, \dots, c_n)$ to a trivial knot.

Theorem 2.6. *For 2-bridge knot/proper link K with Conway's notation $C(c_1, c_2, \dots, c_n)$,*

$$u_R(K) \leq \left\lfloor \frac{\sum_{i \notin L} c_i + 2}{4} \right\rfloor.$$

PROOF. Since any integer is either $\equiv -2, -1, 0$ or $1 \pmod{4}$, we can say that $\sum_{\substack{i < j \\ i \notin L}} c_i \equiv k_j \pmod{4}$ where for each j ($1 \leq j \leq n$), $k_j = -2, -1, 0$ or 1 . Note that for each $j \in L$, after making region crossing changes at $R_{i,3-k_i}; R_{i,7-k_i}; \dots; R_{i,3+4 \cdot \lfloor \frac{c_i-2}{4} \rfloor - k_i}$ regions in each c_i (for $i < j$, $i \notin L$ and $c_i + k_i \geq 2$), c_j can be untangle by just simple twists. Hence for each $j \notin L$, if $c_j + k_j \geq 2$ then region crossing changes at $R_{j,3-k_j}; R_{j,7-k_j}; \dots; R_{j,3+4 \cdot \lfloor \frac{c_j-2}{4} \rfloor - k_j}$ regions in c_j of $C(c_1, c_2, \dots, c_n)$ results a diagram of a trivial knot. \square

Suppose the 2-bridge knot with Conway's notation $C(c_1, c_2, \dots, c_n)$ satisfies

1. c_{2k+1} is *even* for each non-negative integer k such that $2k+1 \leq n$ or
2. c_{2k} is *even* for each positive integer k such that $2k \leq n$ and n is even

then we can provide a better upper bound for region unknotting number. Note that a 2-bridge link $C(c_1, c_2, \dots, c_n)$, where $c_{2k+1} = \text{even}$ for each non-negative integer k such that $2k+1 \leq n$ and n is odd, is not proper iff $\sum_{i=2k+1} c_i \equiv 2 \pmod{4}$.

Theorem 2.7. *For a 2-bridge knot $C(c_1, c_2, \dots, c_n)$*

1. *if c_{2k+1} is even for each non-negative integer k such that $2k+1 \leq n$ then*

$$u_R(K) \leq \left\lfloor \frac{\sum_{i=2k+1} c_i + 2}{4} \right\rfloor,$$

2. if c_{2k} is even for each positive integer k such that $2k \leq n$ and n is even then

$$u_R(K) \leq \min \left\{ \left\lfloor \frac{\sum_{i=2k} c_i + 2}{4} \right\rfloor, \left\lfloor \frac{\sum_{i \notin L} c_i + 2}{4} \right\rfloor \right\}.$$

PROOF. **Case (i)** When c_{2k+1} is even for each non-negative integer k such that $2k+1 \leq n$.

Consider the 2-bridge knot $C(c_1, c_2, \dots, c_n)$, where c_{2k+1} is even. Region crossing changes at $\langle R_3, R_7, \dots, R_{3+4 \cdot \left\lfloor \frac{\sum_{i=2k+1} c_i - 2}{4} \right\rfloor} \rangle$ regions in $C(c_1, c_2, \dots, c_n)$, as in Figure 2, results in a diagram of trivial knot. Observe that the region crossing change at R_j for any $j \in \{c_1 + 1, c_1 + c_2 + 1, \dots, \sum_{i=1}^{\lceil \frac{n-2}{2} \rceil} c_{2i-1} + 1\}$, results in 2 crossing changes in horizontal tangles, one in some $t_{c_{2k-1}}$ and other in $t_{c_{2k+1}}$. Region crossing change at other region R_i will result in 2 crossing changes in some $t_{c_{2k-1}}$. After making above said region crossing changes, the absolute value of sum of signs of all the crossings of horizontal tangles in the resultant diagram becomes zero. Hence

$$u_R(K) \leq \left\lfloor \frac{\sum_{i=2k+1} c_i + 2}{4} \right\rfloor.$$

Case (ii) Proof follows similarly as in Case (i). Here we need to change $\langle R'_3, R'_7, \dots, R'_{3+4 \cdot \left\lfloor \frac{\sum_{i=2k} c_i - 2}{4} \right\rfloor} \rangle$ regions to get a diagram of trivial knot. □

Example 1. Consider the 2-bridge knot $C(2, 3, 4, 2, 6)$. Observe that region crossing changes at $\langle R_3, R_7, R_{11} \rangle$ regions transform $C(2, 3, 4, 2, 6)$ to a diagram of trivial knot as shown in Figure 7. Figure 7(a) is showing 2-bridge knot $C(2, 3, 4, 2, 6)$ and the diagram of trivial knot obtained by above said region crossing changes is shown in Figure 7(b).

Based on the above results, region unknotting number for some 2-bridge knots is provided in Table 1.

3. Arf Invariant of 2-bridge knots

In this section we discuss the Arf invariant of 2-bridge knots. Arf invariant is well defined for knots and proper links in different ways [2, 3, 4]. A relation between Arf invariant and region crossing change is given by Z. Cheng [5]. Let L be a reduced diagram of a proper link and L' is obtained from L by a region crossing change at a region R in L then L' is also proper and there Arf invariants are related by the following theorem:

Theorem 3.1. [5] Let L be a diagram of a proper link, L' is obtained by taking region crossing change on region R of L , where R is white colored region in checkerboard coloring of L , then

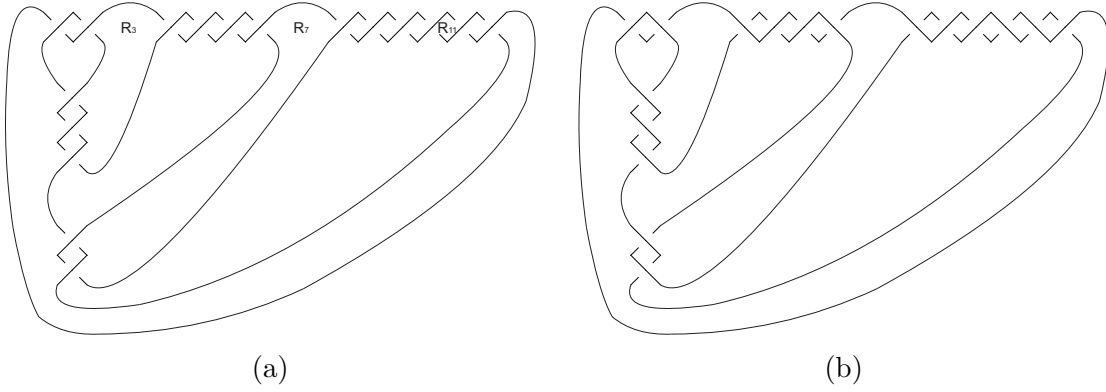


Figure 7: 2-bridge link $C(2, 3, 4, 2, 6)$

Table 1: Region unknotting number of some 2-bridge knots

Knot K	$u_R(K)$	Knot K	$u_R(K)$
$C(2, c_2, 2, c_4)$	1	$C(c_1, 2, c_3, 2)$	1
$C(6, c_2, 2)$	≤ 2	$C(4, c_2, 2, c_4, 2)$	≤ 2
$C(4, c_2, 4)$	≤ 2	$C(2, c_2, 4, c_4, 2)$	≤ 2
$C(2, c_2, 6)$	≤ 2	$C(2, c_2, 2, c_4, 4)$	≤ 2
$C(4, c_2, 2)$	≤ 2	$C(2, c_2, 2, c_4, 2)$	≤ 2
$C(2, c_2, 4)$	≤ 2	$C(4, c_2, 2, c_4, 2, c_6)$	≤ 2
$C(6, c_2, 2, c_4)$	≤ 2	$C(2, c_2, 4, c_4, 2, c_6)$	≤ 2
$C(4, c_2, 4, c_4)$	≤ 2	$C(2, c_2, 2, c_4, 4, c_6)$	≤ 2
$C(2, c_2, 6, c_4)$	≤ 2	$C(2, c_2, 2, c_4, 2, c_6)$	≤ 2
$C(4, c_2, 2, c_4)$	≤ 2	$C(2, c_2, 2, c_4, 2, c_6, 2)$	≤ 2
$C(2, c_2, 4, c_4)$	≤ 2	$C(2, c_2, 2, c_4, 2, c_6, 2, c_8)$	≤ 2

$$\text{Arf}(L) + \text{Arf}(L') = \begin{cases} 0 \pmod{2} & \text{if } \frac{1}{2} \sum_{i=1}^m (a(c_i) - w(c_i)) \equiv 0 \pmod{4}; \\ 1 \pmod{2} & \text{if } \frac{1}{2} \sum_{i=1}^m (a(c_i) - w(c_i)) \equiv 2 \pmod{4}. \end{cases}$$

Here $\{c_1, c_2, \dots, c_m\}$ denote the crossing points on the boundary of R and $a(c_i)$ and $w(c_i)$ are defined as in Figure 8. \square

Also we can use this to calculate Arf invariant of a knot or proper link. Assuming unbounded region as white colored, if we denote

$$A(R) = \begin{cases} \frac{1}{2} \sum_{i=1}^m (a(c_i) - w(c_i)) & \text{if } R \text{ is white colored;} \\ -\frac{1}{2} \sum_{i=1}^m (a(c_i) + w(c_i)) & \text{if } R \text{ is black colored} \end{cases}$$

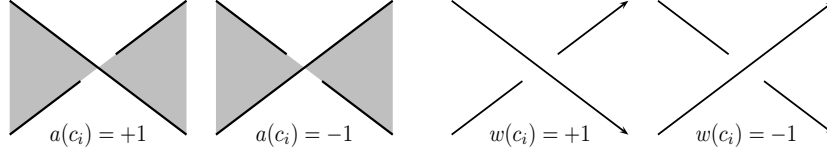


Figure 8: $a(c_i)$ and $w(c_i)$



Figure 9: Parallel and opposite oriented tangles

then Z. Cheng proved the following in [5]:

Theorem 3.2. [5] *Let L be a reduced diagram of a proper link, R_1, \dots, R_n some regions of L , such that region crossing changes at R_1, \dots, R_n will turn L to be trivial. Then,*

$$\text{Arf}(L) = \begin{cases} 0 & \text{if } \sum_{i=1}^n A(R_i) \equiv 0 \pmod{4}; \\ 1 & \text{if } \sum_{i=1}^n A(R_i) \equiv 2 \pmod{4}. \end{cases}$$

□

We use Theorem 3.2 to calculate Arf invariant of different 2-bridge knots. In a 2-bridge knot $C(c_1, c_2, \dots, c_n)$, we call a 2-tangle a parallel oriented tangle or an opposite oriented tangle based on whether both the strands are parallel or opposite oriented as in Figure 9. Observe that for any crossing c in horizontal tangle t_{c_i} , $w(c) = -1$ or 1 based on whether t_{c_i} is parallel or opposite oriented. Similarly for any crossing c in vertical tangle t'_{c_i} , $w(c) = 1$ or -1 based on whether t'_{c_i} is parallel or opposite oriented. The values of $w(c)$ are shown in Table 2. Also $a(c)$ for any crossing c in 2-bridge knot is -1 .

Table 2: Value of $w(c)$ in different conditions

	parallel oriented	opposite oriented
horizontal tangle (t_{c_i})	-1	1
vertical tangle (t'_{c_i})	1	-1

Note that Arf invariant of a link depends on orientation. Let L be an oriented link and link L' is obtained from L after changing the orientation of one of its component. Observe that Arf invariants of L and L' may differ since smoothing of crossings according to orientation in L and L' may result with knots having different Arf invariants. Dependence of Arf invariant on the orientation of links

is shown in Example 2. However, Arf invariant of a knot remains same on reversing the orientation of knot. We calculate Arf invariant for some 2-bridge knot (not link) classes assuming unbounded region as white colored as in Figure 2. Observe that $C(m, n)$ is a link diagram if both m and n are odd. In the following theorem, we calculate Arf invariant for 2-bridge knot (not link) whose Conway's notation is $C(m, n)$.

Theorem 3.3. *Consider a 2-bridge knot with Conway's notation $C(m, n)$*

1. *when only one of either m or n is even. Specifically, if n is even and*
 - (a) $n \equiv 0 \pmod{4}$, then

$$\text{Arf}(C(m, n)) = \begin{cases} 0 & \text{if } \frac{n}{2} \equiv 0 \pmod{4}; \\ 1 & \text{if } \frac{n}{2} \equiv 2 \pmod{4}. \end{cases}$$

- (b) $n \equiv 2 \pmod{4}$, then

$$\text{Arf}(C(m, n)) = \begin{cases} 0 & \text{if } m + \frac{n}{2} \equiv 0 \pmod{4}; \\ 1 & \text{if } m + \frac{n}{2} \equiv 2 \pmod{4}. \end{cases}$$

2. *when both m and n are even and if*
 - (a) $n \equiv 0 \pmod{4}$, then

$$\text{Arf}(C(m, n)) = 0$$

- (b) $n \equiv 2 \pmod{4}$, then

$$\text{Arf}(C(m, n)) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod{4}; \\ 1 & \text{if } m \equiv 2 \pmod{4}. \end{cases}$$

PROOF. **Case (i)** Observe that in $C(m, n)$, the horizontal tangle t_m is opposite oriented and the vertical tangle t'_n is parallel oriented. Hence at each crossing c , we have $w(c) = 1$ and $a(c) = -1$. Since the positions of regions R_i which turn $C(m, n)$ to unknot, is provided in Theorem 2.1, it is easy to calculate $A(R_i)$ for each of these regions R_i . In particular for any R_i ,

$$A(R_i) = -(\text{number of crossings on } \partial R_i).$$

Hence

$$\sum A(R_i) = \begin{cases} -\frac{n}{2} & \text{if } n \equiv 0 \pmod{4}; \\ -(m + \frac{n}{2}) & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Since $\sum_i A(R_i)$ is always even, by Theorem 3.2, the result follows.

Case (ii) When both m and n are even then both t_m and t'_n tangles are opposite oriented. Hence for each crossing c in t_m , $w(c) = 1$ and for each crossing c in t'_n , $w(c) = -1$. Observe that all the regions R_i provided by Theorem 2.1, whose change transform $C(m, n)$ into trivial knot, are white colored. Hence,

$$\sum A(R_i) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{4}; \\ -m & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

Now by Theorem 3.2, the result follows. □

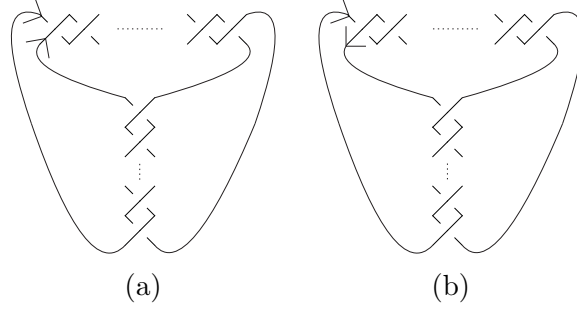


Figure 10: 2-bridge link $C(m, n)$ with different orientations

Example 2. Arf invariant of 2-bridge proper link $C(m, n)$ shown in Figure 10(a) and 10(b) is not same. For Figure 10(a)

$$\text{Arf}(C(m, n)) = \begin{cases} 0 & \text{if } 2 \lfloor \frac{m+2}{4} \rfloor \equiv 0 \pmod{4}; \\ 1 & \text{if } 2 \lfloor \frac{m+2}{4} \rfloor \equiv 2 \pmod{4}; \end{cases}$$

and for Figure 10(b)

$$\text{Arf}(C(m, n)) = \begin{cases} 0 & \text{if } 2 \lfloor \frac{n+2}{4} \rfloor \equiv 0 \pmod{4}; \\ 1 & \text{if } 2 \lfloor \frac{n+2}{4} \rfloor \equiv 2 \pmod{4}; \end{cases}$$

respectively. Note that in case of Figure 10(a), $a(c) = -1 = w(c)$ for each crossing c and in case of Figure 10(b), $a(c) = -1 = -w(c)$. Also $\sum_{i=1}^k A(R_i) = 2 \lfloor \frac{m+2}{4} \rfloor$ and $-2 \lfloor \frac{n+2}{4} \rfloor$ for links in Figure 10(a) and 10(b), respectively. By Theorem 2.1, let region crossing change at R_i ($i = 1, 2, \dots, k$) transforms $C(m, n)$ to a trivial link. For example the Arf invariant for 2-bridge link $C(7, 5)$ is 0 or 1 when orientation is taken as in Figure 10(a) or 10(b) respectively.

Note that $C(m, p, n)$ will be a knot (not link) if and only if either only one of m or n is even or all m, p and n are odd.

Theorem 3.4. Consider a 2-bridge knot with Conway's notation $C(m, p, n)$

1. when only one of either m or n is even. Specifically, if m is even

$$\text{Arf}(C(m, p, n)) = \begin{cases} 0 & \text{if } \sum_i A(R_i) \equiv 0 \pmod{4}; \\ 1 & \text{if } \sum_i A(R_i) \equiv 2 \pmod{4}, \end{cases}$$

where if

(a) p is even

$$\sum_i A(R_i) = \begin{cases} 2 \lfloor \frac{m+n+2}{4} \rfloor & \text{if } m \equiv 0 \pmod{4}; \\ p + 2 \lfloor \frac{m+n+2}{4} \rfloor & \text{if } m \equiv 2 \pmod{4}. \end{cases}$$

(b) p is odd

$$\sum_i A(R_i) = \begin{cases} \lfloor \frac{m}{2} \rfloor & \text{if } m \equiv 0 \pmod{4}; \\ p + \lfloor \frac{m}{2} \rfloor & \text{if } m \equiv 2 \pmod{4}. \end{cases}$$

2. Otherwise, when all m , p and n are odd and if

(a) $p + n \equiv 0 \pmod{4}$, then

$$\text{Arf}(C(m, p, n)) = \begin{cases} 0 & \text{if } 2\lfloor \frac{p+2}{4} \rfloor \equiv 0 \pmod{4}; \\ 1 & \text{if } 2\lfloor \frac{p+2}{4} \rfloor \equiv 2 \pmod{4}. \end{cases}$$

(b) $p + n \equiv 2 \pmod{4}$, then

$$\text{Arf}(C(m, p, n)) = \begin{cases} 0 & \text{if } m + 1 + 2\lfloor \frac{p}{4} \rfloor \equiv 0 \pmod{4}; \\ 1 & \text{if } m + 1 + 2\lfloor \frac{p}{4} \rfloor \equiv 2 \pmod{4}. \end{cases}$$

PROOF. Note that when m , p are even and n is odd then $a(c) = -1 = w(c)$ for each crossing c in $C(m, p, n)$. When m is even and n , p are odd and if crossing c is in t_m or t'_p then $a(c) = -1 = w(c)$, otherwise if crossing c is in t_n then $a(c) = -1 = -w(c)$. Also when all m , p and n are odd, $a(c) = -1 = -w(c)$ for all crossings c in $C(m, p, n)$. Theorem 2.3 and Theorem 2.5 provide the positions of region crossing changes to transform $C(m, p, n)$ to a trivial knot. Further result is based on Theorem 3.2 and simple calculation of $\sum_i A(R_i)$. \square

References

- [1] Ayaka Shimizu, Region crossing change is an unknotting operation, to appear in *J. Math. Soc. Japan*, math.GT/1011.6304v3 (2012).
- [2] Raymond A. Robertello, An invariant of knot cobordism, *Communications on Pure and Applied Mathematics*. **18** (1965) 543–555.
- [3] Murasugi K., The Arf Invariant for Knot Types, *Proceedings of the American Mathematical Society*. **21** (1969) 69–72.
- [4] Kauffman, Louis H., On knots. Annals of Mathematics Studies 115. Princeton University Press (1987) ISBN 0-691-08435-1.
- [5] Cheng Zhiyun, When is region crossing change an unknotting operation?, *Math. Proc. Cambridge Philos. Soc.* **155** (2012) 257–269.
- [6] V. Siwach and P. Madeti, A sharp upper bound for region unknotting number of torus knots, *J. Knot Theory Ramifications*. **22** (2013) 1–21.
- [7] Y. Nakanishi, Replacements in the Conway third identity, *Tokyo J. Math.* **14** (1991) 197–203.
- [8] Haruko Aida, Unknotting operation for Polygonal type, *Tokyo J. Math.* **15** (1992) 111–121.
- [9] J. Hoste, Y. Nakanishi and K. Taniyama, Unknotting operations involving trivial tangles, *Osaka J. Math.* **27** (1990) 555–566.

- [10] H. Murakami, Y. Nakanishi, On a certain move generating link-homology, *Math. Ann.* **284** (1989) 75–89.
- [11] H. Murakami, Some metrics on classical knots, *Math. Ann.* **270** (1985) 35–45.
- [12] R. E. Goodrick, Two bridge knots are alternating knots, *Pacific J. Math.* **40** (1972) 561–564.
- [13] Menasco, W. and Thistlethwaite, M., The classification of alternating links, *Ann. of Math.* **138** (1993) 113–171.
- [14] Schubert, H., Über eine numerische Knoteninvariante, *Math. Zeitschrift.* **61**, (1954) 245–288.